Matrix Decompositions and Quantum Circuit Design

Stephen S. Bullock
(joint with Vivek V.Shende,Igor L.Markov, U.M. EECS)

Mathematical and Computational Sciences Division
National Institute of Standards and Technology

Second **F**eynman **F**estival
University of Maryland College Park
August 21, 2004

Motivation

Classical Problem: Design `AND-OR-NOT` circuit for $\varphi : (\mathbb{F}_2)^n \to \mathbb{F}_2$, with $\mathbb{F}_2 = \{0,1\}$

One Answer: (see e.g. *Feynman on Computation*, section 2.4) Wire an `AND` circuit for each bit string on which $\varphi = 1$; connect circuit blocks by `OR`'s

- Restatement:
  - Produce a decomposition of the function φ
  - Produce circuit blocks accordingly

Matrix decompositions: decompose unitary matrices,
e.g. quantum computations

Motivation, Cont.

However, the approach described here is so simple and general that it does not need an expert in logic to design it! Moreover, it is also a standard type of layout that can easily be laid out in silicon. (ibid.)

Remarks:

- Analog for quantum computers?
- Simple & general?

Outline

I. Two Qubit Circuits (CD)
II. Optimal Relative Phase Circuits
III. Half CNOT per Entry (CSD)
IV. Differntial Topology & Lower Bounds

The Magic Basis of Two-Qubit State Space

$$\begin{cases} |\mathsf{m0}\rangle &= (|00\rangle + |11\rangle)/\sqrt{2} \\ |\mathsf{m1}\rangle &= (|01\rangle - |10\rangle)/\sqrt{2} \\ |\mathsf{m2}\rangle &= (i|00\rangle - i|11\rangle)/\sqrt{2} \\ |\mathsf{m3}\rangle &= (i|01\rangle + i|10\rangle)/\sqrt{2} \end{cases}$$

Remark: Bell states up to global phase; global phases needed for theorem

Theorem (Lewenstein, Kraus, Horodecki, Cirac 2001)
Consider a 4×4 unitary u, global-phase chosen for $\det(u) = 1$

- Compute matrix elements in the magic basis
- (All matrix elements are real) $\Longleftrightarrow$ $(u = a \otimes b)$

Two-Qubit Canonical Decomposition

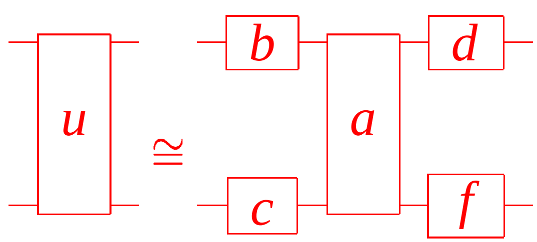
Two-Qubit Canonical Decomposition: Any u a four by four unitary admits a matrix decomposition of the following form:

$$u = (d \otimes f)a(b \otimes c)$$

for $b \otimes c, d \otimes f$ are tensors of one-qubit computations, $a = \sum_{j=0}^{3} \mathrm{e}^{i\theta_j} |\mathrm{mj}\rangle\langle\mathrm{mj}|$

Note that a applies relative phases to the magic or Bell basis.

Circuit diagram: For any u a two-qubit computation, we have:

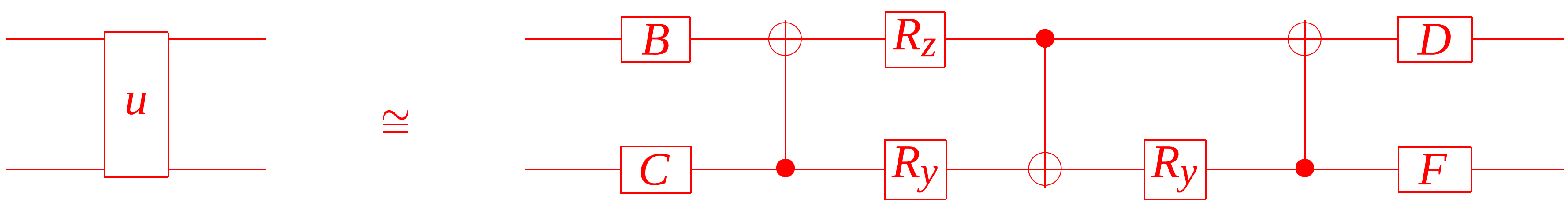


Application: Three CNOT Universal Two-Qubit Circuit

- **Many groups: 3 CNOT circuit for 4×4 unitary:**
  (F.Vatan, C.P.Williams), (G.Vidal, C.Dawson), (V.Shende, I.Markov, B-)

  – Implement a somehow, commute SWAP through circuit to cancel

  – Earlier B-,Markov: 4 CNOT circuit w/o SWAP, CD & naïve a

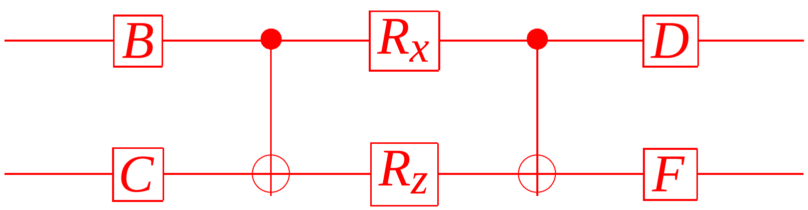


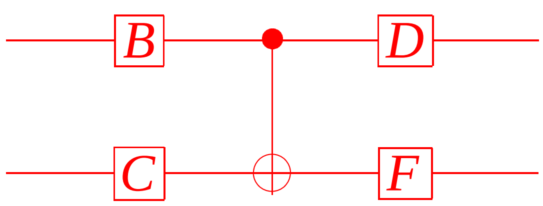
Two-Qubit CNOT-Optimal Circuits

Theorem:(Shende,B-,Markov) Suppose u is a 4×4 unitary normalized so $\det(u) = 1$. Label $\gamma(v) = (-i\sigma^y)^{\otimes 2} v (-i\sigma^y)^{\otimes 2} v^T$. Then any v admits a circuit holding elements of $SU(2)^{\otimes 2}$ and 3 CNOT's, up to global phase. Moreover, for $p(\lambda) = \det[\lambda I_4 - \gamma(v)]$ the characteristic poly of $\gamma(v)$:

- (v admits a circuit with 2 CNOT's) $\Longleftrightarrow$ ($p(\lambda)$ has real coefficients)
- (v admits a circuit with 1 CNOT) $\Longleftrightarrow$ ($p(\lambda) = (\lambda + i)^2 (\lambda - i)^2$)
- ($v \in SU(2) \otimes SU(2)$) $\Longleftrightarrow$ ($\gamma(v) = \pm I_4$)

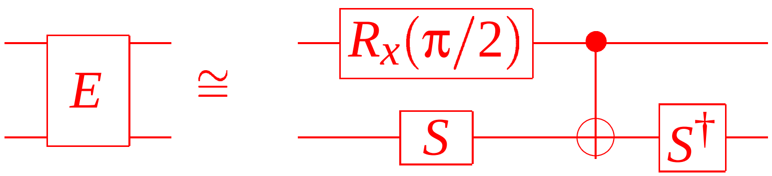
Optimal Structured Two-qubit Circuits





- Quantum circuit identities: All $1,2$ CNOT diagrams reduce to these

- Computing parameters: useful to use operator E, $E|j\rangle = |\mathrm{mj}\rangle$



Outline

I. Two Qubit Circuits (CD)
II. Optimal Relative Phase Circuits
III. Half CNOT per Entry (CSD)
IV. Differntial Topology & Lower Bounds

Relative Phase Group

- Easiest concievable n-qubit circuit question: How to build circuits for

$$A(2^n) = \left\{ \sum_{j=0}^{2^n-1} \mathrm{e}^{i\theta_j} |j\rangle\langle j| \; ; \; \theta_j \in \mathbb{R} \right\}?$$

- $A(2^N)$ commutative $\Longrightarrow$ vector group
  - $\log A(2^n) \to \mathfrak{a}(2^n)$ carries matrix multiplication to vector sum
  - Strategy: build decompositions from vector space decompositions
  - Subspaces encoded by characters, i.e. continuous group maps $\chi : A(2^n) \to U(1)$

Characters Detecting Tensors

- ker $\log \chi$ is a subspace of $\mathfrak{a}(2^n)$

- Subspaces $\bigcap_j$ ker $\log \chi_j$ exponentiate to closed subgroups

Example: $a = \sum_{j=0}^{2^n-1} z_j |j\rangle\langle j| \in A(2^n)$ has $a = \tilde{a} \otimes R_z(\alpha)$ if and only if

$$z_0/z_1 = z_2/z_3 = \cdots = z_{2^n-2}/z_{2^n-1}$$

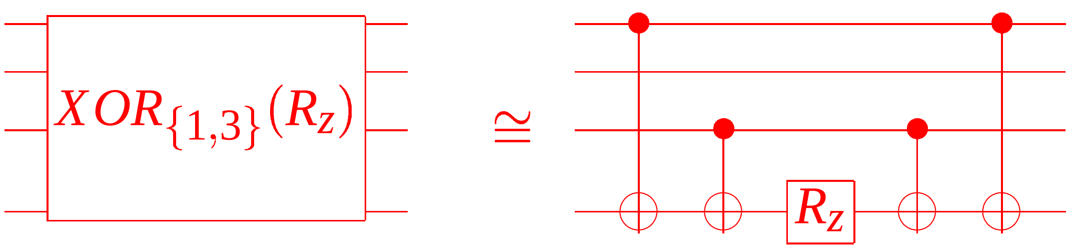
So a factors on the bottom line if and only if $a \in \bigcap_{j=0}^{2^{n-1}-1}$ ker χ_j
for $\chi_j(a) = z_{2j} z_{2j+2}/(z_{2j+1} z_{2j+3})$.

Circuits for $A(2^n)$

Outline of Synthesis for $A(2^n)$:

- Produce circuit blocks capable of setting all $\chi_j = 1$
- After $a = \tilde{a} \otimes R_z$, induct to $\tilde{a}$ on top $n-1$ lines

Remark: $2^{n-1} - 1$ characters to zero $\Longrightarrow 2^{n-1} - 1$ blocks, i.e. one for each nonempty subset of the top $n-1$ lines



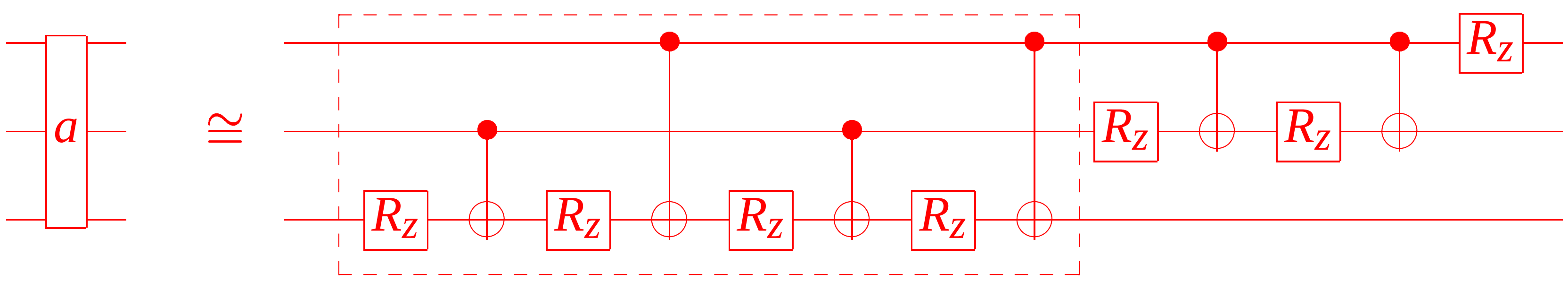
Circuits for $A(2^n)$, Cont.

Tricks in Implementing Outline:

- If $\#[(S_1 \cup S_2) - (S_1 \cap S_2)] = 1$, then all but one CNOT in center of $XOR_{S_1}(R_z)\, XOR_{S_2}(R_z)$ cancel

- Take subsets in Gray code, most CNOTs cancel

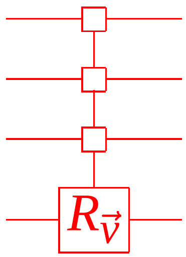
- Final count: $2^n - 2$ CNOTs



Uniformly Controlled Rotations
(M.Möttönen, J.Vartiainen)

Let $\vec{v}$ be any axis on Block sphere. Uniformly-controlled rotation requires 2^{n-1} CNOTs:

$$\bigwedge_k^{\text{uni}}[R_{\vec{v}}] = \begin{pmatrix} R_{\vec{v}}(\theta_0) & \mathbf{0}_2 & \cdots & \mathbf{0}_2 \\ \mathbf{0}_2 & R_{\vec{v}}(\theta_1) & \cdots & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \ddots & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \cdots & R_{\vec{v}}(\theta_{2^{n-1}-1}) \end{pmatrix}$$



Example: Outlined block is diag$[R_z(\theta_1), R_z(\theta_2), \cdots, R_z(\theta_{2^{n-1}})] = \bigwedge_{n-1}^{\text{uni}}[R_z]$ up to SWAP of qubits $1,n$

Shende, q-ph/0406176: Short proof of 2^{n-1} CNOTs using induction:
$\mathfrak{a}(2^n) = I_2 \otimes \mathfrak{a}(2^{n-1}) \oplus \sigma^z \otimes \mathfrak{a}(2^{n-1})$

Outline

I. Two Qubit Circuits (CD)
II. Optimal Relative Phase Circuits
III. Half CNOT per Entry (CSD)
IV. Differntial Topology & Lower Bounds

Universal Circuits

Goal: Build a universal quantum circuit for u be $4^n \times 4^n$ unitary evolution

- Change rotation angles: any u up to phase
- Preview: At least $4^n - 1$ rotation boxes $R_{\vec{v}}$, at least $\frac{1}{4}(4^n - 3n - 1)$ CNOTs
- Prior art
  - Barenco Bennett Cleve DiVincenzo Margolus Shor Sleator J.Smolin Weinfurter (1995) $\approx 50n^2 \times 4^n$ CNOTs
  - Vartiainen, Möttönen, Bergholm, Salomaa, $\approx 8 \times 4^n$ (2003), $\approx 4^n$ (2004)

Cosine Sine Decomposition

Cosine Sine Decomposition: Any v a $2^n \times 2^n$ unitary may be written

$$v = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix} = (a_1 \oplus b_1)\gamma(a_2 \oplus b_2)$$

where a_j, b_j are $2^{n-1} \times 2^{n-1}$ unitary, $c = \sum_{j=0}^{2^{n-1}-1} \cos t_j |j\rangle\langle j|$ and $s = \sum_{j=0}^{2^{n-1}-1} \sin t_j |j\rangle\langle j|$

- Studied extensively in numerical matrix analysis literature

- Fast CSD algorithms exist; reasonable on laptop for $n = 10$

Strategy for $\approx 4^n/2$ CNOT Circuit

- Use CSD for $v = (a_1 \oplus b_1)\gamma(c_1 \oplus d_1)$
- Implement $\gamma = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ as uniformly controlled rotations
  - uniform control $\Longrightarrow$ few CNOTs
- Implement $a_j \oplus b_j = \begin{pmatrix} a_j & 0 \\ 0 & b_j \end{pmatrix}$ as quantum multiplexor
  - Also includes uniformly controlled rotations, also inductive
- Induction ends at specialty two-qubit circuit

Quantum Multiplexors

- Multiplexor: route computation as control bit $0,1$

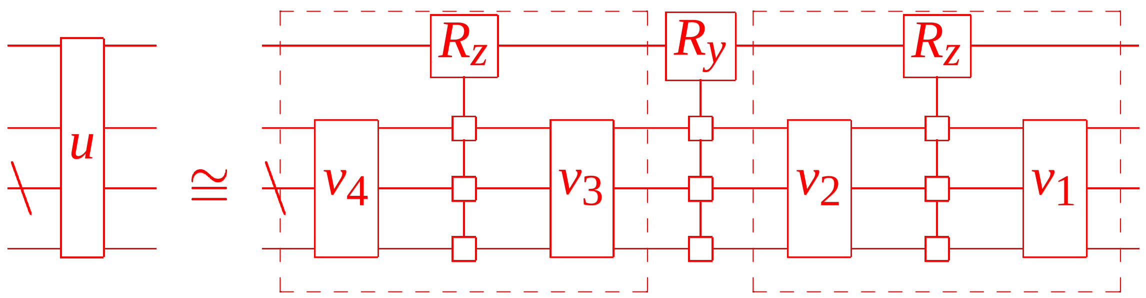
- $v = a \oplus b$: Do a or b as top qubit $|0\rangle$, $|1\rangle$

- Diagonalization trick: Solve following system, $d \in A(2^{j-1})$, u,w each some $2^{n-1} \times 2^{n-1}$ unitary

$$\begin{cases} a &= udw \\ b &= ud^{\dagger}w \end{cases}$$

- Result: $a \oplus b = (u \oplus u)(d \oplus d^{\dagger})(w \oplus w) = (I_2 \otimes u) \wedge_{n-1}^{\text{uni}}[R_z](I_2 \otimes w)$

Circuit for $(1/2)$ CNOT per Entry



- Outlined sections are multiplexor implementations

- Cosine Sine matrix γ: uniformly controlled $\wedge_{n-1}^{\text{uni}}[R_y]$
  - Only 2^{n-1} CNOTs, converts to R_z by conjugation by HS

Circuit Errata

- Lower bound $\Longrightarrow$ (can be improved by no more than factor of 2)

- 21 CNOTs in 3 qubits: currently best known

- $\approx 50\%$ CNOTs on bottom two lines
  - Adapts to spin-chain architecture with $(4.5) \times 4^n$ CNOTs
  - Quantum charge couple device (QCCD) with 3 or 4 qubit chamber?

Outline

I. Two Qubit Circuits (CD)
II. Optimal Relative Phase Circuits
III. Half CNOT per Entry (CSD)
IV. Differntial Topology & Lower Bounds

Sard's Theorem

Def: A critical value of a smooth function of smooth manifolds $f : M \to N$ is any $n \in N$ such that there is some $p \in M$ with $f(p) = n$ with the linear map $(df)_p : T_pM \to T_nN$ not onto.

Sard's theorem: The set of critical values of any smooth map has measure zero.

Corollary: If dim $M <$ dim N, then image(f) is measure 0.

- $U(2^n) = \{u \in \mathbb{C}^{2^n \times 2^n}\,;\, uu^\dagger = I_{2^n}\}$: smooth manifold

- Circuit topology τ with k one parameter rotation boxes induces smooth evaluation map $f_\tau : U(1) \times \mathbb{R}^k \to U(2^n)$

Dimension-Based Bounds

- Consequence: Any universal circuit must contain $4^n - 1$ one parameter rotation boxes

- No consolidation: Boxes separated by at least $\frac{1}{4}(4^n - 3n - 1)$ CNOTs
  - v Bloch sphere rotation: $v = R_x R_z R_x$ or $v = R_z R_x R_z$
  - Diagrams below: consolidation if fewer CNOTs

On-going Work

- Subgroups H of unitary group $U(2^n)$
  - More structure, smaller circuits?
  - Symmetries encoded within subgroups H
  - Native gate libraries?
- Special purpose circuits
  - Backwards: quantum circuits for doing numerical linear algebra?
  - Entanglement dynamics and circuit size

`http://www.arxiv.org` **Coordinates**

- Two-qubits: `q-ph/0308045`
- Diagonal circuits: `q-ph/0303039`
- Uniform control: `q-ph/0404089`
- $(1/2)$ CNOT/entry: `q-ph/0406176`
- Circuit diagrams by `Qcircuit.tex`: `q-ph/0406003`